

A NEW CHARACTERIZATION OF AMENABLE GROUPS¹

BY

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ABSTRACT. Paradoxical sets, which are a natural generalization of the type of sets made famous as Hausdorff-Banach-Tarski paradoxes, are defined in terms of piecewise translations. Piecewise translations are the generalization to arbitrary discrete groups of the maps used in the Banach-Tarski paradoxes as congruences by finite decomposition. A subset of a group is defined to be large if finitely many translates of it can cover the group. The main result of this paper is that a group is amenable if and only if it does not contain a large paradoxical set.

1. Introduction. A Banach measure on a group G is a finitely additive, nonnegative, translation invariant measure which is defined for every subset of G . A group on which a Banach measure can be defined is called amenable.

Stefan Banach [1] showed that \mathbf{R} , the additive group of real numbers, is amenable. von Neumann [2] observed that the same proof shows that any Abelian group is amenable. Hausdorff [3] constructed subsets of the 2-sphere which are simultaneously one-half the sphere and one-third the sphere. Starting from this paradox, Banach and Tarski [4] showed that any two bounded subsets of E^3 with nonempty interiors are congruent by finite decomposition. Two sets A and B are defined to be congruent by finite decomposition if each can be written as the disjoint union of n subsets such that the i th subset of A is congruent to the i th subset of B for $i = 1, 2, \dots, n$. It is clear that if A and B are subsets of an amenable group G and are congruent by finite decomposition, then the measures of A and B are equal.

Thus Banach and Tarski showed that a Banach measure cannot be defined on the group of isometries of E^3 . von Neumann [2] pointed out that these odd sets constructed by Hausdorff, Banach, and Tarski depend on the existence in the group of isometries of E^3 of a free subgroup on two generators. The natural question then is: if a group G does not contain a free subgroup on two generators, is G amenable? This question is yet unanswered. Another question one might ask is: does every group which fails to be amenable

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contain a paradoxical set of the Banach-Tarski type, or can amenability fail in some other way? We prove in this paper that if the notion of paradoxical set is suitably generalized, then a group is amenable if and only if it does not contain a "large" paradoxical set. The word large is needed here, since there are amenable groups with paradoxical sets which can consistently be assigned measure zero.

2. Piecewise translations and paradoxical sets.

DEFINITION 2.1. A piecewise translation of a subset A of a group G is a function $f: A \rightarrow f(A)$ such that

$$A = \bigcup_{i=1}^n A_i, \quad \text{where } A_i \cap A_j = \emptyset \quad \text{if } i \neq j;$$

and

$$\begin{aligned} f(a) &= \alpha_i a \quad \text{for all } a \in A_i, \\ \alpha_i A_i \cap \alpha_j A_j &= \emptyset \quad \text{if } i \neq j. \end{aligned}$$

We have identified piecewise translations with translations on the left. To be completely general we should also define right piecewise translations and two-sided piecewise translations. In Definitions 2.4 and 2.8 we will define paradoxical sets in terms of piecewise translations and proceed with the proof of the main result of this paper: that if a group does not contain such a paradoxical set, then it is amenable. Since not containing a paradoxical set by left piecewise translations is a weaker hypothesis than not containing a paradoxical set by general piecewise translations (left, right, or two-sided), we prove a slightly stronger theorem as well as gain some simplicity by working only with left piecewise translations.

THEOREM 2.2. (1) *A piecewise translation is a 1-to-1 function.*

(2) *If $f: A \rightarrow B$ is a piecewise translation, then $f^{-1}: B \rightarrow A$ is also.*

(3) *The composition of piecewise translations is again a piecewise translation.*

(4) *The relation $B = f(A)$, where f is a piecewise translation, is an equivalence relation on the subsets of G .*

(5) *If $B = f(A)$, where f is a piecewise translation, then A and B have the same measure on any Banach measure defined on G .*

(6) *If f is a piecewise translation of S , then its restriction to a subset $T \subset S$ is a piecewise translation of T .*

PROOF. These statements are all obvious.

DEFINITION 2.3. A set $S \subset G$ is *large* if there exist piecewise translations f_1, f_2, \dots, f_n such that

$$G = \bigcup_{i=1}^n f_i(S).$$

It is easy to see that this is equivalent to the existence of finitely many elements x_1, x_2, \dots, x_m of G such that the translates $x_i S$ cover G (i.e., $G \subset \bigcup_{i=1}^m x_i S$).

It also follows easily from this definition that if $A \subset B$ and A is large then B is large.

DEFINITION 2.4. A set $S \subset G$ is $n + 1$ -into- n if there exist piecewise translations f_1, f_2, \dots, f_{n+1} and g_1, \dots, g_n such that

$$\bigcup_{i=1}^{n+1} f_i(S) \subset \bigcup_{i=1}^n g_i(S)$$

and

$$f_i(S) \cap f_j(S) = \emptyset \quad \text{for } i \neq j.$$

DEFINITION 2.5. A set $E \subset G$ is a Δ -set of the set $S \subset G$ if there is a piecewise translation f such that $f(S) \subset S$ and $E = S - f(S)$.

THEOREM 2.6. *The following conditions are equivalent:*

- (1) G contains an $n + 1$ -into- n .
- (2) G contains a 2-into-1.
- (3) G contains a set S and a Δ -set E of S such that $S \subset \bigcup_{i=1}^n f_i(E)$, where f_1, \dots, f_n are piecewise translations.

THEOREM 2.7. *The following conditions are equivalent:*

- (1) G contains a large $n + 1$ -into- n .
- (2) G contains a large 2-into-1.
- (3) G contains a large Δ -set.

PROOF OF THEOREMS 2.6 AND 2.7. We will prove that $3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 3$ for both Theorems 2.6 and 2.7.

$3 \Rightarrow 2$. Assume $E = S - f(S)$, where $f(S) \subset S$ and $S \subset \bigcup_{i=1}^n g_i(E)$, where f and g_1, g_2, \dots, g_n are piecewise translations. Then $f^2(S) \subset f(S)$. Next let $E_2 = f(S) - f^2(S)$. In this way define $E_i = f^{i-1}(S) - f^i(S)$ for $i = 1, \dots, n$. Let $E_1 = E$. Then $E_i \cap E_j = \emptyset$ if $i \neq j$, and

$$\bigcup_{i=1}^n E_i \subset \bigcup_{i=1}^n [f^{i-1}(S) - f^i(S)] \subset S - f^n(S).$$

Since $S \subset \bigcup_{i=1}^n g_i(E)$ and $E_i = f^{i-1}(E_1)$, it follows that we can define a piecewise translation h such that $h(S) \subset \bigcup_{i=1}^n E_i$. Then $f^n(S) \cup h(S) \subset S$ and $f^n(S) \cap h(S) = \emptyset$. Thus S is a 2-into-1.

If E is large, then, since $E \subset S$, S is large, and thus S is a large 2-into-1.

$2 \Rightarrow 1$. This is true because a 2-into-1 is by definition an $n + 1$ -into- n .

$1 \Rightarrow 3$. Suppose that $\bigcup_{i=1}^{n+1} f_i(S) \subset \bigcup_{i=1}^n g_i(S)$, where $f_i(S) \cap f_j(S) = \emptyset$ if $i \neq j$ and the f_i and g_i are piecewise translations.

Let $S' = \bigcup_{i=1}^n g_i(S)$ and define a piecewise translation $h: S' \rightarrow S'$ by

$$\begin{aligned} h(g_1(S)) &= f_1 g_1^{-1}(g_1(S)) = f_1(S), \\ h(g_2(S) - g_1(S)) &= f_2 g_2^{-1}(g_2(S) - g_1(S)) \subset f_2(S), \\ h(g_3(S) - g_2(S) - g_1(S)) &= f_3 g_3^{-1}(g_3(S) - g_2(S) - g_1(S)) \subset f_3(S), \\ &\vdots \\ h(g_n(S) - g_{n-1}(S) - \cdots - g_1(S)) \\ &= f_n g_n^{-1}(g_n(S) - g_{n-1}(S) - \cdots - g_1(S)) \subset f_n(S). \end{aligned}$$

Then

$$h(S') \subset \bigcup_{i=1}^n f_i(S) \subset \bigcup_{i=1}^{n+1} f_i(S) \subset \bigcup_{i=1}^n g_i(S)$$

and

$$E = S' - h(S') \supset f_{n+1}(S).$$

It is easy to see that $S' \subset \bigcup_{i=1}^n g_i f_{n+1}^{-1}(f_{n+1}(S))$, and since $f_{n+1}(S) \subset E$ it follows that there exist piecewise translations e_1, \dots, e_n such that $S' \subset \bigcup_{i=1}^n e_i(E)$. We already know that if A is large and $A \subseteq B$ then B is large. Thus if we are given that S is large, then it follows that E is large, since $f_{n+1}(S) \subset E$. \square

DEFINITION 2.8. We say that a group G contains a large *paradoxical set* if for some cyclic group C_n the direct sum $G \times C_n$ contains a large 2-into-1.

THEOREM 2.9. If G contains a large paradoxical set, then G is not amenable.

PROOF. If S is a large 2-into-1 in $G \times C_n$, then $f_1(S) \cup f_2(S) \subset g(S)$ and $f_1(S) \cap f_2(S) = \emptyset$ implies that $2\mu(S) \leq \mu(S)$ for any Banach measure μ . Thus $\mu(S) = 0$. But since S is large, $G \times C_n \subset \bigcup_{i=1}^m h_i(S)$. Thus $m \cdot \mu(S) > 1$.

This contradiction shows that $G \times C_n$ is not amenable. Since it is well known that the group $A \times B$ is amenable if A and B are, and that C_n is amenable, it follows that G is not amenable. \square

3. Definition of upper and lower measures.

DEFINITION 3.1. If $A, B \subset G$, then we say $m \times A \subset n \times B$ if there exist piecewise translations f_1, f_2, \dots, f_m and g_1, \dots, g_n in $G \times C_s$ for some sufficiently large s such that

$$\bigcup_{i=1}^m f_i(A) \subset \bigcup_{i=1}^n g_i(B) \subset G \times C_s$$

and

$$f_i(A) \cap f_j(A) = \emptyset \quad \text{if } i \neq j.$$

THEOREM 3.2. *If $A, B, C \subset G$, then:*

(1) *If $n_1 \times A \subset m_1 \times B$ and $n_2 \times A \subset m_2 \times B$, then*

$$(n_1 + n_2) \times A \subset (m_1 + m_2) \times B.$$

(2) *If $n \times A \subset m \times B$, then $pn \times A \subset pm \times B$.*

(3) *If $n \times A \subset m \times B$ and $m \times B \subset r \times C$, then $n \times A \subset r \times C$.*

(4) *If $n \times A_1 \subset m_1 \times B$ and $n \times A_2 \subset m_2 \times B$, then*

$$n \times (A_1 \cup A_2) \subset (m_1 + m_2) \times B.$$

(5) *If $n_1 \times A \subset m \times B_1$ and $n_2 \times A \subset m \times B_2$ and $B_1 \cap B_2 = \emptyset$, then*

$$(n_1 + n_2)A \subset m \times (B_1 \cup B_2).$$

PROOF. These are all obvious consequences of Definition 3.1.

DEFINITION 3.3. If $A \subset G$ we say $A \geq 1/n$ if $G \subset n \times A$. Equivalently, $A \geq 1/n$ means there exist n piecewise translations f_1, \dots, f_n such that $G \subset \bigcup_{i=1}^n f_i(A)$.

DEFINITION 3.4. If $A \subset G$ we define the set of positive numbers L_A as follows:

$$a \in L_A \text{ if } a = 1/n_1 + 1/n_2 + \dots + 1/n_k$$

and

$$A = \bigcup_{i=1}^k A_i, \quad \text{where } A_i \cap A_j = \emptyset \quad \text{if } i \neq j \quad \text{and} \quad A_i \geq 1/n_i.$$

We call L_A the set of lower bounds of A .

THEOREM 3.5. *If G contains no large paradoxical sets, then L_A is bounded above and $\sup L_A \leq 1$ for every $A \subset G$.*

PROOF. Assume $a \in L_A$ for $A \subset G$ and that $a > 1$. Then $a = 1/n_1 + \dots + 1/n_k$ and $A = \bigcup_{i=1}^k A_i$, where $A_i \geq 1/n_i$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. Then

$$G \subset n_i \times A_i \quad \text{for } i = 1, 2, \dots, k. \quad (1)$$

Let $p = n_1 \cdot n_2 \cdot \dots \cdot n_k$. Then $pa = p/n_1 + \dots + p/n_k > p$. We are going to show that if s is any positive integer greater than pa , then $G \times C_s$ contains a large $n+1$ -into- n . Therefore choose a positive integer $s > pa$. Then by hypothesis, $G \times C_s$ contains no large 2-into-1. From (1) and Theorem 3.2, it follows that $p/n_i \times G \subset p \times A_i$ for $i = 1, 2, \dots, k$. Thus, by Theorem 3.2, we have $(\sum_{i=1}^k p/n_i) \times G \subset p \times A$. But $\sum_{i=1}^k p/n_i = pa$. Thus $pa \times G \subset p \times A$, and since $p \times A \subset p \times G$, we have $pa \times G \subset p \times G$, where $pa > p$.

Therefore G is a large $n + 1$ -into- n in $G \times C_s$, and it follows from Theorem 2.7 that $G \times C_s$ contains a large 2-into-1. This contradicts our assumption that G contains no large paradoxical set and shows that $\sup L_A \leq 1$ for every $A \subset G$. \square

It is interesting to note that if G contains a large paradoxical set, then $\sup L_A$ is unbounded for all large sets. To prove this, note that a group G contains a large 2-into-1 if and only if G is a large 2-into-1. Then if A is a large subset of G and $G \times C_n$ is a large 2-into-1 for some n , then A covers $G \times C_n$. But the same number of images of A also cover $2 \times (G \times C_n)$ and $4 \times (G \times C_n)$ and $8 \times (G \times C_n)$, etc.

DEFINITION 3.6. If G does not contain a large paradoxical set, we define a lower measure $\underline{\mu}(A) = \sup L_A$ for every $A \subset G$, where $\underline{\mu}(A) = 0$ if $L_A = \emptyset$.

THEOREM 3.7. If G is a group which does not contain a large paradoxical set, then the lower measure $\underline{\mu}$ satisfies the following conditions:

- (1) $\underline{\mu}(A) \leq 1$ for every $A \subset G$.
- (2) $\underline{\mu}(f(A)) = \underline{\mu}(A)$ for every $A \subset G$ and every piecewise translation f .
- (3) $\underline{\mu}(G) = 1$.
- (4) $\underline{\mu}(A \cup B) \geq \underline{\mu}(A) + \underline{\mu}(B)$ for $A, B \subset G$ with $A \cap B = \emptyset$.

PROOF. Statement (1) is a consequence of Theorem 3.5. Since $A \geq 1/n$ means that $G = \bigcup_{i=1}^n f_i(A)$, where f_1, \dots, f_n are piecewise translations, it follows that $g(A) \geq 1/n$ for any piecewise translation g . Thus if $a \in L_A$, then $a \in L_{g(A)}$. This proves (2).

To show (3) we observe that $G = G$. Thus $1 \in L_G$, and from Theorem 3.5 we have $\sup L_G \leq 1$. Thus $\sup L_G = 1$.

To prove (4), suppose $\alpha \in L_A$ and $\beta \in L_B$. Then $A = \bigcup_{i=1}^k A_i$, where $A_i \geq 1/n_i$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\alpha = \sum_{i=1}^k 1/n_i$; and $B = \bigcup_{i=1}^l B_i$, where $B_i \geq 1/m_i$ and $B_i \cap B_j = \emptyset$ if $i \neq j$ and $\beta = \sum_{i=1}^l 1/m_i$. Then $A \cup B = \bigcup_{i=1}^k A_i \cup (\bigcup_{i=1}^l B_i)$.

Thus $\alpha + \beta \in L_{A \cup B}$ and $\sup L_{A \cup B} \geq \alpha + \beta$. Since α and β are arbitrary elements of L_A and L_B , it follows that

$$\sup L_{A \cup B} \geq \sup L_A + \sup L_B. \quad \square$$

DEFINITION 3.8. If $A \subset G$ we say that $A \leq 1/n$ if $n \times A \subset G$. $A \leq 1/n$ is equivalent to the existence of n piecewise translations f_1, \dots, f_n such that $\bigcup_{i=1}^n f_i(A) \subset G$ and $f_i(A) \cap f_j(A) = \emptyset$ if $i \neq j$.

DEFINITION 3.9. If $A \subset G$ we associate with A the set of positive numbers U_A , the set of upper bounds of A : $U_A = \{\alpha: \alpha = \sum_{i=1}^k 1/n_i \text{ and } A = \bigcup_{i=1}^k A_i, \text{ where } A_i \leq 1/n_i\}$.

DEFINITION 3.10. If G is a group which does not contain a large paradoxical set, then we define an upper measure $\bar{\mu}(A) = \inf U_A$, for every $A \subset G$.

THEOREM 3.11. *If G is a group which does not contain a large paradoxical set, then the upper measure $\bar{\mu}$ satisfies the following conditions:*

- (1) $\bar{\mu}(G) = 1$.
- (2) $\bar{\mu}(f(A)) = \bar{\mu}(A)$ for every $A \subset G$ and every piecewise translation f .
- (3) $\bar{\mu}(A \cup B) \leq \bar{\mu}(A) + \bar{\mu}(B)$ for $A, B \subset G$ with $A \cap B = \emptyset$.
- (4) $\bar{\mu}(A) \leq \bar{\mu}(B)$ if $A \subset B$.

PROOF. (1) Suppose there is an $\alpha \in U_G$ with $\alpha < 1$. Then $G = \bigcup_{i=1}^k A_i$ with $A_i \leq 1/n_i$ and $\alpha = \sum_{i=1}^k 1/n_i < 1$. Thus $n_i \times A_i \subset G$ for $i = 1, \dots, k$. Let $P = n_1 \cdot n_2 \cdot n_3 \cdot \dots \cdot n_k$. Then, by Theorem 3.2,

$$((P/n_i) \cdot n_i) \times A_i \subset P/n_i \times G \quad \text{for } i = 1, 2, \dots, k.$$

Thus

$$P \times A_i \subset P/n_i \times G \quad \text{for } i = 1, 2, \dots, k.$$

Then

$$P \times \bigcup_{i=1}^k A_i \subset \sum_{i=1}^k P/n_i \times G$$

follows from Theorem 3.2.

Since $\bigcup_{i=1}^k A_i = G$ and $\sum_{i=1}^k P/n_i = P\alpha < P$, this implies $P \times G \subset P\alpha \times G$, where $P > P\alpha$.

Thus G is a large $n+1$ -into- n in $G \times C_s$ for some s . Then we know from Theorem 2.7 that $G \times C_s$ contains a large 2-into-1 and therefore G contains a large paradoxical set. This contradiction shows that for every $\alpha \in U_G$, $\alpha \geq 1$. Since $G \subset G$, we have $1 \in U_G$; thus $\inf U_G = 1$ and $\bar{\mu}(G) = 1$.

The proof of (2) is a simple consequence of the fact that the property $A \leq 1/n$ is by definition invariant under piecewise translation.

To prove (3), assume that $\alpha \in U_A$ and $\beta \in U_B$. Then $A = \bigcup_{i=1}^k A_i$, where $A_i \leq 1/n_i$ and $B = \bigcup_{i=1}^t B_i$, where $B_i \leq 1/m_i$, and $\alpha = \sum_{i=1}^k 1/n_i$ and $\beta = \sum_{i=1}^t 1/m_i$. Then

$$A \cup B = \left(\bigcup_{i=1}^k A_i \right) \cup \left(\bigcup_{i=1}^t B_i \right),$$

where $A_i \leq 1/n_i$ and $B_i \leq 1/m_i$. Thus

$$\sum_{i=1}^k \frac{1}{n_i} + \sum_{i=1}^t \frac{1}{m_i} = \alpha + \beta \in U_{A+B}.$$

Thus $\inf U_{A+B} \leq \alpha + \beta$. Since α and β were arbitrary elements of U_A and U_B , it follows that $\inf U_{A+B} \leq \inf U_A + \inf U_B$.

To prove (4) we first observe that if $A_i \subset B_i$ and $B_i \leq 1/n_i$, then $A_i \leq 1/n_i$. Now suppose $\alpha \in U_B$ and $A \subset B$. Then $B = \bigcup_{i=1}^k B_i$, where $B_i \leq 1/n_i$

and $\alpha = \sum_{i=1}^k 1/n_i$. Then let $A_i = A \cap B_i$. Hence $A_i \leq 1/n_i$ and $A = \bigcup_{i=1}^k A_i$; therefore $\alpha \in U_A$, and so $\inf U_A \leq \inf U_B$. \square

One may observe that if G contains a large 2-into-1, then $\inf U_G = 0$.

THEOREM 3.12. *If G does not contain a large paradoxical set, then $\underline{\mu}(A) \leq \bar{\mu}(A)$ for every $A \subset G$.*

The point of this theorem is to show that if G does not contain large paradoxical sets, then there is always room between the lower and upper measures to assign a Banach measure. The upper and lower measures of a subset of G are forced by the conditions of finite additivity and translation invariance, and if there is no room between them, then amenability fails. If in fact G contains a large 2-into-1, then this does fail in the wildest possible way, since $\underline{\mu}(G) = \infty > \bar{\mu}(G) = 0$.

PROOF. Suppose $\sup L_A > \inf U_A$. Then there exist $\alpha = \sum_{i=1}^k 1/n_i \in L_A$ and $\beta = \sum_{j=1}^t 1/m_j \in U_A$, with $\alpha > \beta$. Then $A = \bigcup_{i=1}^k A_i$, $A_i \cap A_j = \emptyset$ if $i \neq j$; and $A = \bigcup_{j=1}^t B_j$ with $G \subset n_i \times A_i$ for $i = 1, \dots, k$ and $m_j \times B_j \subset G$ for $j = 1, \dots, t$. Let $Q = \prod_{i=1}^k n_i$ and $P = \prod_{j=1}^t m_j$. Then $((QP/m_j) \cdot m_j) \times B_j \subset QP/m_j \times G$ for $j = 1, \dots, t$; and $QP/n_i \times G \subset ((QP/n_i) \cdot n_i) \times A_i$ for $i = 1, \dots, k$. Thus $QP/n_i \times G \subset QP \times A_i$ for $i = 1, \dots, k$; and $QP \times B_j \subset QP/m_j \times G$ for $j = 1, \dots, t$.

Thus $(\sum_{i=1}^k QP/n_i) \times G \subset QP \times A$ and $QP \times A \subset (\sum_{j=1}^t QP/m_j) \times G$. Then $QP\alpha \times G \subset QP \times A$ and $QP \times A \subset QP\beta \times G$. And so $QP\alpha \times G \subset QP\beta \times G$. If $\alpha > \beta$, this implies that G contains a large paradoxical set. Thus $\alpha \leq \beta$ and $\sup L_A \leq \inf U_A$ and $\underline{\mu}(A) \leq \bar{\mu}(A)$. \square

4. Rational sets.

DEFINITION 4.1. If G is a group with no large paradoxical sets, then a subset $A \subset G$ is a *rational set* if $\underline{\mu}(A) = \bar{\mu}(A)$.

For rational sets A we set $\mu(A) = \underline{\mu}(A) = \bar{\mu}(A)$.

THEOREM 4.2. *If A and B are rational subsets of a group G with no large paradoxical set and $A \cap B = \emptyset$, then $A \cup B$ is a rational set and $\mu(A \cup B) = \mu(A) + \mu(B)$.*

PROOF. $\mu(A \cup B) \geq \underline{\mu}(A) + \underline{\mu}(B) = \bar{\mu}(A) + \bar{\mu}(B) \geq \bar{\mu}(A \cup B)$. Thus $\mu(A \cup B) \geq \bar{\mu}(A \cup B)$. But $\underline{\mu}(A \cup B) \leq \bar{\mu}(A \cup B)$. Therefore $\mu(A \cup B) = \bar{\mu}(A \cup B)$ and $A \cup B$ is a rational set with $\mu(A \cup B) = \mu(A) + \mu(B)$. \square

THEOREM 4.3. *If A is a rational set, then so is $f(A)$ for any piecewise translation f , and $\mu(f(A)) = \mu(A)$.*

PROOF. This follows from Theorems 3.11 and 3.7.

THEOREM 4.4. *On the collection of rational subsets of G , where G contains no large paradoxical set, the set function μ is finitely additive and piecewise translation invariant with $\mu(G) = 1$.*

PROOF. This is a consequence of Theorems 3.7, 3.11, 4.2, and 4.3.

5. Rational functions and nonnegative linear functionals. We have now shown that if G is a group without large paradoxical sets then we can define a finitely additive, piecewise translation invariant set function μ on the collection of rational subsets of G such that $\mu(\emptyset) = 0$ and $\mu(G) = 1$. (It is clear that \emptyset and G are rational sets.) We now wish to extend this set function to a measure on every subset of G . For the rational sets $A \subset G$ we have $\mu(A) = \bar{\mu}(A)$, and thus there is no choice but to choose $\mu(A) = \bar{\mu}(A)$. However, if A is not rational, then $\underline{\mu}(A) < \bar{\mu}(A)$, and we have infinitely many choices for $\mu(A)$. To make these choices consistently we wish to use a variation of the Hahn-Banach extension theorem. To that end we make the following definitions:

DEFINITION 5.1. Let V be the vector space generated by characteristic functions χ_A of subsets $A \subset G$. Then, if $f \in V$,

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i},$$

where $\alpha_i \in \mathbf{R}$ and $A_i \subset G$.

Let M be the subspace of V generated by the characteristic functions of rational subsets of G . If $f \in M$, then

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i},$$

where $\alpha_i \in \mathbf{R}$, and A_i are rational subsets of G .

DEFINITION 5.2. We want to define a linear functional λ on M as follows. If $f \in M$ and

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i},$$

where A_i is a rational subset of G for $i = 1, 2, \dots, n$, then

$$\lambda(f) = \sum_{i=1}^n \alpha_i \cdot \mu(A_i).$$

We need to show that this is a proper definition, i.e., that $\sum_{i=1}^n \alpha_i \cdot \mu(A_i)$ is well defined. The remainder of this section will be devoted to proving that we can define a linear functional this way.

DEFINITION 5.3. A *piecewise isomorphism* is a one-to-one function $\varphi: G \xrightarrow{\text{onto}} G'$ such that if f is a piecewise translation of a subset $A \subset G$, then $f'(x) = \varphi f \varphi^{-1}(x)$ is a piecewise translation of $\varphi(A) \subset G'$.

THEOREM 5.4. *If $\varphi: G \rightarrow G'$ is a piecewise isomorphism, and G' is amenable, then G is amenable.*

PROOF. Suppose m' is a Banach measure on G' . Then define $m(A) = m'(\varphi(A))$ for every $A \subset G$. Then m is a Banach measure on G :

$$(1) m(G) = m'(\varphi(G)) = m'(G') = 1.$$

(2) If $A \cap B = \emptyset$, then

$$m(A \cup B) = m'(\varphi(A \cup B)) = m'(\varphi(A) \cup \varphi(B)) = m'(\varphi(A)) + m'(\varphi(B)),$$

since $\varphi(A) \cap \varphi(B) = \emptyset$.

(3) If f is a piecewise translation,

$$m(f(A)) = m'(\varphi(f(A))) = m'(f'(\varphi(A))) = m'(\varphi(A)) = m(A),$$

where $f' = \varphi f \varphi^{-1}$ is a piecewise translation of $\varphi(A) \subset G'$. \square

THEOREM 5.5. *If $\varphi: G \rightarrow G'$ is a piecewise isomorphism and G contains a 2-into-1 (resp., large 2-into-1), then G' contains a 2-into-1 (resp., large 2-into-1).*

PROOF. Suppose $f_1(A) \cup f_2(A) \subset A$ and $f_1(A) \cap f_2(A) = \emptyset$, where f_1 and f_2 are piecewise translations. Then $f'_1 = \varphi f_1 \varphi^{-1}$ and $f'_2 = \varphi f_2 \varphi^{-1}$ are piecewise translations, and

$$f'_1(\varphi(A)) \cup f'_2(\varphi(A)) \subset \varphi A \quad \text{and} \quad f'_1(\varphi(A)) \cap f'_2(\varphi(A)) = \emptyset.$$

If A is large, then $G \subset \bigcup_{i=1}^n g_i(A)$, where the g_i are piecewise translations. Then $G' \subset \bigcup_{i=1}^n g'_i(\varphi(A))$ and the $g'_i = \varphi g_i \varphi^{-1}$ are piecewise translations. Thus $\varphi(A)$ is large in G' . \square

THEOREM 5.6. *If $\varphi: G \rightarrow G'$ and $\psi: H \rightarrow H'$ are piecewise isomorphisms, then so is $\varphi \times \psi: G \times H \rightarrow G' \times H'$ defined by*

$$\varphi \times \psi(g \cdot h) = \varphi(g)\psi(h)$$

for every $gh \in G \times H$, with $g \in G$, $h \in H$.

PROOF. If $f: S \rightarrow f(S)$ is a piecewise translation on $G \times H$, then $S = \bigcup_{i=1}^n A_i$ and $f(S) = \bigcup_{i=1}^n \alpha_i A_i$. Each $\alpha_i = \beta_i \varepsilon_i$ where $\beta_i \in G$ and $\varepsilon_i \in H$. Then let $f': \varphi \times \psi(S) \rightarrow \varphi \times \psi(f(S))$ be defined by

$$f'(x) = \varphi \times \psi \cdot f \cdot (\varphi \times \psi)^{-1}(x)$$

for every $x \in S' = \varphi \times \psi(S)$. If $(a, b) = (\varphi \times \psi)^{-1}(x) \in G \times H$, then

$$f'(x) = \varphi \times \psi \cdot f(a, b) = \varphi \times \psi(\beta_i a, \varepsilon_i b) = \varphi(\beta_i a)\psi(\varepsilon_i b).$$

The 1-to-1 function $f'(x)$ is a piecewise translation on $S' \subset G' \times H'$ if and only if the set of translates $\{f'(x) \cdot x^{-1}: x \in S'\}$ is finite. This is just the set

$$\begin{aligned} & \{ \varphi(\beta_i a) \psi(\varepsilon_i b) \cdot [\varphi \times \psi(ab)]^{-1} : ab \in S \} \\ &= \{ \varphi(\beta_i a) [\varphi(a)]^{-1} \psi(\varepsilon_i b) [\psi(b)]^{-1} : ab \in S \}. \end{aligned}$$

Let

$$\pi_G(A_i) = \{ a \in G : (a, b) \in A_i \}$$

and

$$\pi_H(A_i) = \{ b \in H : (a, b) \in A_i \}.$$

Define a translation f_i on $\pi_G(A_i)$ by $f_i(a) = \beta_i a$; and a translation h_i on $\pi_H(A_i)$ by $h_i(b) = \varepsilon_i b$. Then both $\{ \varphi f_i(a) \cdot [\varphi(a)]^{-1} : a \in \pi_G(A_i) \}$ and $\{ \psi h_i(b) \cdot [\psi(b)]^{-1} : b \in \pi_H(A_i) \}$ are finite or, equivalently, $\{ \varphi(\beta_i a) \cdot [\varphi(a)]^{-1} : a \in \pi_G(A_i) \}$ and $\{ \psi(\varepsilon_i b) \cdot [\psi(b)]^{-1} : b \in \pi_H(A_i) \}$ are finite for $i = 1, 2, \dots, n$.

Then

$$\bigcup_{i=1}^n \{ \varphi(\beta_i a) \cdot [\varphi(a)]^{-1} : a \in \pi_G(A_i) \} = \{ \varphi(\beta_i a) \cdot [\varphi(a)]^{-1} : a \in \pi_G(S) \}$$

and

$$\bigcup_{i=1}^n \{ \psi(\varepsilon_i b) \cdot [\psi(b)]^{-1} : b \in \pi_H(A_i) \} = \{ \psi(\varepsilon_i b) \cdot [\psi(b)]^{-1} : b \in \pi_H(S) \}$$

are finite. Thus

$$\begin{aligned} & \left| \{ \varphi(\beta_i a) [\varphi(a)]^{-1} \psi(\varepsilon_i b) [\psi(b)]^{-1} : ab \in S \} \right| \\ & \leq \left| \{ \varphi(\beta_i a) [\varphi(a)]^{-1} : a \in \pi_G(S) \} \right| \\ & \quad \cdot \left| \{ \psi(\varepsilon_i b) \cdot [\psi(b)]^{-1} : b \in \pi_H(S) \} \right| < \infty. \end{aligned}$$

Thus f' is a piecewise translation of $\varphi \times \psi(S)$ to $\varphi \times \psi(fS)$ in $G' \times H'$. Since it is clear that $\varphi \times \psi$ is 1-to-1 and onto, it follows that $\varphi \times \psi$ is a piecewise isomorphism. \square

COROLLARY 5.6'. *If φ is a piecewise isomorphism from G to G' and h is any 1-to-1 function from the finite group A onto the finite group B , then $\varphi_h: G \times A \rightarrow G' \times B$ defined by $\varphi_h(ga) = \varphi g \cdot ha$ is a piecewise isomorphism.*

PROOF. Since on a finite group any 1-to-1 function is a piecewise translation, it follows that any 1-to-1 function from one finite group onto another finite group is a piecewise isomorphism. Then this is just a special case of Theorem 5.6. \square

COROLLARY 5.7. *If $\varphi: G \rightarrow G'$ is a piecewise isomorphism and G contains a (large) paradoxical set, then G' contains a (large) paradoxical set.*

PROOF. This is a consequence of Theorems 5.5 and 5.6 and the definitions of paradoxical and large paradoxical sets.

LEMMA 5.8. *If G is a group without large paradoxical sets, then for any positive integer s we can define a piecewise translation invariant, finitely additive set function m on the rational subsets of $G \times C_s$ such that if $A \subset G$ is a rational set in G , then $m(A) = \mu(A)$.*

PROOF. Since G contains no large paradoxical set, it follows that $G \times C_s$ contains no large paradoxical set, because a large 2-into-1 in $G \times C_s \times C_n$ implies the existence of a large 2-into-1 in $G \times C_{s \cdot n}$. This follows from Corollary 5.6', since $C_s \times C_n$ and $C_{s \cdot n}$ are of the same order.

Then we can define a piecewise translation invariant, finitely additive set function m^* on the rational subsets of $G \times C_s$. G is a rational subset of $G \times C_s$, and if A is a rational subset of G , it is also a rational subset of $G \times C_s$. Also, $m^*(G) = 1/s$ and $m^*(A) = (1/s)\mu(A)$. Then if we define $m(A) = s \cdot m^*(A)$, we are done. \square

LEMMA 5.9. *If there is an $f \in M$ such that*

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} = \sum_{i=1}^m \beta_i \chi_{B_i},$$

with α_i and $\beta_i \in \mathbf{R}$, where A_i and B_i are rational sets, and

$$\sum_{i=1}^n \alpha_i \mu(A_i) \neq \sum_{i=1}^m \beta_i \mu(B_i),$$

then there is an $f' \in M$ such that

$$f' = \sum_{i=1}^n \alpha'_i \chi_{A_i} = \sum_{j=1}^m \beta'_j \chi_{B_j},$$

where the $\alpha'_i, \beta'_j \in \mathbf{Z}$, and

$$\sum_{i=1}^n \alpha'_i \mu(A_i) \neq \sum_{j=1}^m \beta'_j \mu(B_j).$$

PROOF. Let C be a Hamel basis of \mathbf{R} over \mathbf{Q} . Then for each α_i and β_i there is a unique function $\alpha_i(c), \beta_i(c)$ from $C \rightarrow \mathbf{Q}$ which equals 0 except for finitely many c 's such that

$$\alpha_i = \sum_{c \in C} \alpha_i(c) \cdot c \quad \text{and} \quad \beta_i = \sum_{c \in C} \beta_i(c) \cdot c.$$

Then for each $x \in G$ we have

$$f(x) = \sum_{i=1}^n \left(\sum_{c \in C} \alpha_i(c) \cdot c \right) \chi_{A_i}(x) = \sum_{i=1}^m \left(\sum_{c \in C} \beta_i(c) \cdot c \right) \chi_{B_i}(x).$$

Let c_1, c_2, \dots, c_s be the finitely many elements of C such that $\alpha_i(c) \neq 0$ or $\beta_i(c) \neq 0$ for some i . Then

$$\sum_{i=1}^n \sum_{j=1}^s \alpha_i(c_j) \cdot c_j \cdot \chi_{A_i}(x) = \sum_{i=1}^m \sum_{j=1}^s \beta_i(c_j) \cdot c_j \cdot \chi_{B_i}(x).$$

From the linear independence of C it follows that

$$\sum_{i=1}^n \alpha_i(c_j) \cdot \chi_{A_i}(x) = \sum_{i=1}^m \beta_i(c_j) \cdot \chi_{B_i}(x).$$

Moreover, if

$$\sum_{i=1}^n \alpha_i(c_j) \mu(A_i) = \sum_{i=1}^m \beta_i(c_j) \mu(B_i),$$

for each $j = 1, \dots, s$, then

$$\sum_{j=1}^s \sum_{i=1}^n \alpha_i(c_j) \cdot c_j \cdot \mu(A_i) = \sum_{j=1}^s \sum_{i=1}^m \beta_i(c_j) \cdot c_j \cdot \mu(B_i),$$

contradicting the hypothesis of the lemma.

Therefore, for some j ,

$$\sum_{i=1}^n \alpha_i(c_j) \mu(A_i) \neq \sum_{i=1}^m \beta_i(c_j) \mu(B_i),$$

while

$$\sum_{i=1}^n \alpha_i(c_j) \chi_{A_i} = \sum_{i=1}^m \beta_i(c_j) \chi_{B_i},$$

and the $\alpha_i(c_j)$ and $\beta_i(c_j)$ are rational. We can now multiply through by the common denominator of the $\alpha_i(c_j)$ and the $\beta_i(c_j)$ to get the desired relations with integral coefficients. \square

LEMMA 5.10. *If G is a group with no large paradoxical sets and*

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} = \sum_{i=1}^m \beta_i \chi_{B_i}$$

is a real-valued rational function on G (A_i, B_i are rational sets) such that

$$\sum_{i=1}^n \alpha_i \mu(A_i) \neq \sum_{i=1}^m \beta_i \mu(B_i),$$

then there is a rational function

$$f' = \sum_{i=1}^{n'} \alpha'_i \chi_{A'_i} = \sum_{i=1}^{m'} \beta'_i \chi_{B'_i}$$

such that

$$\sum_{i=1}^{n'} \alpha'_i \mu(A'_i) \neq \sum_{i=1}^{m'} \beta'_i \mu(B'_i)$$

and the α_i and β_i are positive integers.

PROOF. According to the preceding lemma, we may assume that the α_i and β_i are integers. Suppose α_i , $i = r + 1, \dots, n$, and β_i , $i = s + 1, \dots, m$, are negative. Define

$$f' = \sum_{i=1}^r \alpha_i \chi_{A_i} - \sum_{i=s+1}^m \beta_i \chi_{B_i} = \sum_{i=1}^s \beta_i \chi_{B_i} - \sum_{i=r+1}^n \alpha_i \chi_{A_i}.$$

Then f' is a rational function with positive integer coefficients; if

$$\sum_{i=1}^r \alpha_i \mu(A_i) - \sum_{i=s+1}^m \beta_i \mu(B_i) = \sum_{i=1}^s \beta_i \mu(B_i) - \sum_{i=r+1}^n \alpha_i \mu(A_i),$$

then

$$\sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^m \beta_i \mu(B_i). \quad \square$$

THEOREM 5.11. *If G is a group without large paradoxical sets and*

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} = \sum_{i=1}^m \beta_i \chi_{B_i}$$

is a rational function, where A_i and B_i are rational sets, then

$$\sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^m \beta_i \mu(B_i).$$

PROOF. It follows from the preceding lemma that we may assume the α_i and β_i are positive integers. Now choose

$$s > \sum_{i=1}^n \alpha_i + \sum_{i=1}^m \beta_i$$

and define the graph of f , $\text{Gr}(f) \subset G \times C_s$, by

$$\text{Gr}(f) = \{(x, r) \in G \times C_s : 0 < r \leq f(x)\}.$$

Then by Lemma 5.8 there is a measure m on the rational sets of $G \times C_s$ such that $m(A) = \mu(A)$ for rational subsets A of G . We will show that

$$m(\text{Gr}(f)) = \sum_{i=1}^n \alpha_i \mu(A_i).$$

From the definition of $\text{Gr}(f)$ it follows that we can define piecewise translations h_{ij} for $i = 1, \dots, n$ and $j = 1, \dots, \alpha_i$ such that

$$\text{Gr}(f) = \bigcup_{i=1}^n \bigcup_{j=1}^{\alpha_i} h_{ij}(A_i)$$

and $h_{ij}(A_i) \cap h_{i'j'}(A_{i'}) = \emptyset$ unless $i = i'$ and $j = j'$. Then

$$\begin{aligned} m(\text{Gr}(f)) &= \sum_{i=1}^n \sum_{j=1}^{\alpha_i} m(h_{ij}(A_i)) = \sum_{i=1}^n \sum_{j=1}^{\alpha_i} m(A_i) \\ &= \sum_{i=1}^n \alpha_i m(A_i) = \sum_{i=1}^n \alpha_i \mu(A_i). \end{aligned}$$

The $h_{ij}(A_i)$ are rational in $G \times C_s$, since the A_i are rational in G . If we now repeat the same argument for $f = \sum_{i=1}^m \beta_i \chi_{B_i}$, we get

$$m(\text{Gr}(f)) = \sum_{i=1}^m \beta_i \mu(B_i).$$

Thus

$$\sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^m \beta_i \mu(B_i). \quad \square$$

6. Extension of the linear functional.

DEFINITION 6.1. If $f, g \in V$, then $f \approx g$ if

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} \quad \text{and} \quad g = \sum_{i=1}^n \alpha_i \chi_{h_i(A_i)} + \sum_{j=1}^s \beta_j \chi_{B_j},$$

where the h_i are piecewise translations and the B_j are rational sets with $\mu(B_j) = 0$.

If $f, g \in V$, then $f \sim g$ if

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} \quad \text{and} \quad g = \sum_{i=1}^n \alpha_i \chi_{h_i(A_i)},$$

where the h_i are piecewise translations.

It follows from the properties of piecewise translations that there are equivalence relations.

If $f_1 \approx g_1$ and $f_2 \approx g_2$, then it is clear that $f_1 + f_2 \approx g_1 + g_2$, and $af_1 \approx ag_1$ for any constant a .

Similarly, if $f_1 \sim g_1$ and $f_2 \sim g_2$, then $f_1 + f_2 \sim g_1 + g_2$, and $af_1 \sim ag_1$ for any constant a .

Let V' be the collection of equivalence classes with respect to \approx of V . Then V' is a linear space.

THEOREM 6.2. If $f, g \in M$ and $f \approx g$, then $\lambda(f) = \lambda(g)$.

LEMMA 0. If $f = f_1 + f_2$, where f, f_1, f_2 are finite linear combinations of characteristic functions and are positive integer-valued; and if $\text{Gr}(f_1), \text{Gr}(f_2)$ are rational sets in $G \times C_s$, then

$$m(\text{Gr}(f)) = m(\text{Gr}(f_1)) + m(\text{Gr}(f_2)).$$

PROOF. Since $f = f_1 + f_2$, it follows from the definition of Gr that, for some piecewise translation H of $G \times C_s$,

$$\text{Gr}(f) = \text{Gr}(f_1) \cup H(\text{Gr}(f_2))$$

and

$$\text{Gr}(f_1) \cap H(\text{Gr}(f_2)) = \emptyset.$$

Then

$$m(\text{Gr}(f)) = m(\text{Gr}(f_1)) + m(H(\text{Gr}(f_2)))$$

and

$$m(\text{Gr}(f)) = m(\text{Gr}(f_1)) + m(\text{Gr}(f_2)). \quad \square$$

LEMMA 1. If $f = \sum_{i=1}^m \alpha_i \chi_{A_i} = \sum_{i=1}^p \beta_i \chi_{B_i}$ and

$$g = \sum_{i=1}^m \alpha_i \chi_{h(A_i)} = \sum_{i=1}^r \gamma_i \chi_{C_i},$$

where the γ_i, β_i , are positive integers and the α_i are integers (not necessarily positive), and the B_i and C_i are rational sets, then $\lambda(f) = \lambda(g)$.

PROOF. We proved in Theorem 5.11 that if $f = \sum_{i=1}^p \beta_i \chi_{B_i}$, where the β_i are positive integers and the B_i are rational sets, then

$$\sum_{i=1}^p \beta_i \mu(B_i) = m\text{Gr}(f),$$

where

$$\text{Gr}(f) = \{(x, r) \in G \times C_s : 0 < r \leq f(x)\},$$

and s can be any integer greater than $\max\{f(x) \text{ and } g(x)\}$; and m is a finitely additive, piecewise-translation invariant set function defined on rational subsets of $G \times C_s$, which is identical with μ for rational subsets of G .

The existence of such a set function is proved in Lemma 5.8.

From Theorem 5.11 we also have

$$\sum_{i=1}^r \gamma_i \mu(C_i) = m(\text{Gr}(g)).$$

We will now show that $m(\text{Gr}(f)) = m(\text{Gr}(g))$.

Define

$$f_+ = \sum_{\alpha_i > 0} \alpha_i \chi_{A_i} \quad \text{and} \quad f_- = \sum_{\alpha_i < 0} (-\alpha_i \chi_{A_i}),$$

$$g_+ = \sum_{\alpha_i > 0} \alpha_i \chi_{h(A_i)} \quad \text{and} \quad g_- = \sum_{\alpha_i < 0} (-\alpha_i \chi_{h(A_i)}).$$

Then $f = f_+ - f_-$ and $g = g_+ - g_-$.

For each $i = 1, \dots, m$ such that $\alpha_i < 0$, choose a rational set $A'_i \subset G$ with

$A_i \subset A'_i$. Since G is rational, such a set always exists. Then let

$$f'_- = \sum_{\alpha_i < 0} (-\alpha_i \chi_{A_i}) + \sum_{\alpha_i < 0} (-\alpha_i \chi_{(A'_i - A_i)}) = \sum_{\alpha_i < 0} (-\alpha_i \chi_{A'_i})$$

and let

$$f'_+ = f_+ + \sum_{\alpha_i < 0} (-\alpha_i \chi_{(A'_i - A_i)}) = \sum_{\alpha_i > 0} (\alpha_i \chi_{A_i}) + \sum_{\alpha_i < 0} (-\alpha_i \chi_{(A'_i - A_i)}).$$

Then $f = f'_+ - f'_-$, and since f and f'_- are rational functions, so is f'_+ .

Let

$$g'_- = g_- + \sum_{\alpha_i < 0} (-\alpha_i \chi_{(A'_i - A_i)})$$

and

$$g'_+ = g_+ + \sum_{\alpha_i < 0} (-\alpha_i \chi_{(A'_i - A_i)}).$$

Then $g = g'_+ - g'_-$, and $f'_+ \sim g'_+$ and $f'_- \sim g'_-$.

It can be shown from the definition of \sim that if f_1, f_2 are functions for which Gr is defined and $f_1 \sim f_2$, then $\text{Gr}(f_1)$ is a piecewise translation of $\text{Gr}(f_2)$. Therefore, $\text{Gr}(g'_+)$ is a piecewise translation of $\text{Gr}(f'_+)$ and $\text{Gr}(g'_-)$ is a piecewise translation of $\text{Gr}(f'_-)$.

Thus $\text{Gr}(g'_+)$ and $\text{Gr}(g'_-)$ are rational sets in $G \times C$, since they are piecewise translations of $\text{Gr}(f'_+)$ and $\text{Gr}(f'_-)$.

Then $f'_+ = f + f'_-$ and $g'_+ = g + g'_-$.

By Lemma 0:

$$m(\text{Gr}(f'_+)) = m(\text{Gr}(f)) + m(\text{Gr}(f'_-))$$

and

$$m(\text{Gr}(g'_+)) = m(\text{Gr}(g)) + m(\text{Gr}(g'_-)).$$

Since $\text{Gr}(f'_+)$ and $\text{Gr}(f'_-)$ are piecewise translations of $\text{Gr}(g'_+)$ and $\text{Gr}(g'_-)$,

$$m(\text{Gr}(f'_+)) = m(\text{Gr}(g'_+)) \quad \text{and} \quad m(\text{Gr}(f'_-)) = m(\text{Gr}(g'_-)).$$

Thus

$$m(\text{Gr}(f)) = m(\text{Gr}(g)) \quad \text{and} \quad \lambda(f) = \lambda(g). \quad \square$$

LEMMA 2. If $f = \sum_{i=1}^m \alpha_i \chi_{A_i} = \sum_{i=1}^p \beta_i \chi_{B_i}$ and $g = \sum_{i=1}^m \alpha_i \chi_{h_i(A_i)} = \sum_{i=1}^r \gamma_i \chi_{C_i}$, where the $\alpha_i, \beta_i, \gamma_i$ are integers, and the B_i and C_i are rational sets, then $\lambda(f) = \lambda(g)$.

PROOF. Let

$$f^+ = \sum_{\beta_i > 0} \beta_i \chi_{B_i} \quad \text{and} \quad f^- = \sum_{\beta_i < 0} -\beta_i \chi_{B_i},$$

and

$$g^+ = \sum_{\gamma_i > 0} \gamma_i \chi_{C_i} \quad \text{and} \quad g^- = \sum_{\gamma_i < 0} -\gamma_i \chi_{C_i}.$$

Then $f = f^+ - f^-$ and $g = g^+ - g^-$.

Thus f^+, f^-, g^+ , and g^- are rational functions, and $f^+ - f^- \sim g^+ - g^-$. Then $f^+ + g^- \sim g^+ + f^-$. These are rational functions with positive integer coefficients, and it follows from Lemma 1 that $\lambda(f^+ + g^-) = \lambda(g^+ + f^-)$. Then, since λ is linear, we have $\lambda(f^+) + \lambda(g^-) = \lambda(g^+) + \lambda(f^-)$ and $\lambda(f^+) - \lambda(f^-) = \lambda(g^+) - \lambda(g^-)$ or $\lambda(f) = \lambda(g)$. \square

LEMMA 3. *If $f = \sum_{i=1}^m \alpha_i \chi_{A_i} = \sum_{i=1}^p \beta_i \chi_{B_i}$ and $g = \sum_{i=1}^m \alpha_i \chi_{h(A_i)} = \sum_{i=1}^r \gamma_i \chi_{C_i}$, where the $\alpha_i, \beta_i, \gamma_i$ are rational numbers and B_i and C_i are rational sets, then $\lambda(f) = \lambda(g)$.*

PROOF. If $D = \text{L.C.D.}(\alpha_i, \beta_i, \gamma_i)$, then Df and Dg will satisfy the hypotheses of Lemma 2 and $\lambda(Df) = \lambda(Dg)$. Then $D\lambda(f) = D\lambda(g)$ and $\lambda(f) = \lambda(g)$. \square

LEMMA 4. *If $f, g \in M$ and $f \sim g$, then $\lambda(f) = \lambda(g)$.*

PROOF. Suppose $f = \sum_{i=1}^m \alpha_i \chi_{A_i} = \sum_{i=1}^s \beta_i \chi_{B_i}$ and $g = \sum_{i=1}^s \alpha_i \chi_{h(A_i)} = \sum_{i=1}^r \gamma_i \chi_{C_i}$, where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ and B_i and C_i are rational sets.

Let C be a Hamel basis of \mathbb{R} over \mathbb{Q} . Then for each $\alpha_i, \beta_i, \gamma_i$ there is a unique function $\alpha_i(c), \beta_i(c), \gamma_i(c)$ from $C \rightarrow \mathbb{Q}$ which equals 0 except for finitely many c 's such that

$$\alpha_i = \sum_{c \in C} \alpha_i(c) \cdot c, \quad \beta_i = \sum_{c \in C} \beta_i(c) \cdot c, \quad \gamma_i = \sum_{c \in C} \gamma_i(c) \cdot c.$$

Let c_1, c_2, \dots, c_n be the finitely many elements of C such that $\alpha_i(c) \neq 0$ or $\beta_i(c) \neq 0$ or $\gamma_i(c) \neq 0$, for some i .

Then

$$\alpha_i = \sum_{j=1}^n \alpha_i(c_j) \cdot c_j, \quad \beta_i = \sum_{j=1}^n \beta_i(c_j) \cdot c_j, \quad \gamma_i = \sum_{j=1}^n \gamma_i(c_j) \cdot c_j.$$

Thus

$$f = \sum_{i=1}^m \left(\sum_{j=1}^n \alpha_i(c_j) \cdot c_j \right) \chi_{A_i} = \sum_{i=1}^s \left(\sum_{j=1}^n \beta_i(c_j) \cdot c_j \right) \chi_{B_i},$$

and

$$g = \sum_{i=1}^m \left(\sum_{j=1}^n \alpha_i(c_j) \cdot c_j \right) \chi_{h(A_i)} = \sum_{i=1}^r \left(\sum_{j=1}^n \gamma_i(c_j) \cdot c_j \right) \chi_{C_i}.$$

Then

$$\sum_{j=1}^n \left(\sum_{i=1}^m \alpha_i(c_j) \chi_{A_i} \right) c_j = \sum_{j=1}^n \left(\sum_{i=1}^s \beta_i(c_j) \chi_{B_i} \right) c_j$$

and

$$\sum_{j=1}^n \left(\sum_{i=1}^m \alpha_i(c_j) \chi_{A_i}(c_j) \right) c_j = \sum_{j=1}^n \left(\sum_{i=1}^r \gamma_i(c_j) \chi_{C_i}(c_j) \right) c_j.$$

It follows from the linear independence of C that for each $j = 1, 2, \dots, n$

$$\sum_{i=1}^m \alpha_i(c_j) \chi_{A_i} = \sum_{i=1}^s \beta_i(c_j) \chi_{B_i} \quad (1)$$

and

$$\sum_{i=1}^m \alpha_i(c_j) \chi_{A_i} = \sum_{i=1}^r \gamma_i(c_j) \chi_{C_i}. \quad (2)$$

Let

$$f_j = \sum_{i=1}^m \alpha_i(c_j) \chi_{A_i} \quad \text{and} \quad g_j = \sum_{i=1}^m \alpha_i(c_j) \chi_{A_i}.$$

Then $f_j \sim g_j$, and the coefficients $\alpha_i(c_j)$, $\beta_i(c_j)$, and $\gamma_i(c_j)$ are rational. Further, equations (1) and (2) show that f_j and g_j are rational functions.

Thus by Lemma 3 we have $\lambda(f_j) = \lambda(g_j)$ for all $j = 1, 2, \dots, n$. Then since λ is linear we have

$$\lambda f = \lambda \left(\sum_{j=1}^n c_j f_j \right) = \sum_{j=1}^n c_j \lambda(f_j) = \sum_{j=1}^n c_j \lambda(g_j) = \lambda \left(\sum_{j=1}^n c_j g_j \right) = \lambda g. \quad \square$$

This completes the proof of Lemmas 1-4.

Now suppose that

$$f = \sum_{i=1}^m \alpha_i \chi_{A_i} = \sum_{i=1}^s \beta_i \chi_{B_i}$$

and

$$g = \sum_{i=1}^m \alpha_i \chi_{A_i} + \sum_{i=1}^t \varphi_i \chi_{D_i} = \sum_{i=1}^r \gamma_i \chi_{C_i},$$

where the $\alpha_i, \beta_i, \gamma_i, \varphi_i \in \mathbb{R}$, the B_i, C_i, D_i are rational sets and $\mu(D_i) = 0$.

Then let $g' = g - \sum_{i=1}^t \varphi_i \chi_{D_i}$. By Lemma 4, $\lambda(g') = \lambda(f)$. But

$$\lambda(g') = \lambda g - \lambda \left(\sum_{i=1}^t \varphi_i \chi_{D_i} \right) = \lambda g - \sum_{i=1}^t \varphi_i \mu(D_i) = \lambda g.$$

Thus $\lambda(g) = \lambda(f)$. This completes the proof. \square

DEFINITION 6.3. Define the relation $>$ on V' by defining $F > 0$ if there exists $f \in F$ such that $f > 0$. Then $F_1 > F_2$ if $F_1 - F_2 > 0$.

THEOREM 6.4. If $f > 0$ and $g \leq 0$ and $f \sim g$, then $f = g = 0$.

If $f > 0$ and $g \leq 0$ and $f \approx g$, then $f - g = 0$ for all $x \in G - E$, where $\mu(E) = 0$.

PROOF. Suppose that

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} > 0 \quad \text{and} \quad g = \sum_{i=1}^n \alpha_i \chi_{h(A_i)} < 0,$$

where the α_i are integers.

Then

$$f - g = \sum_{i=1}^n \alpha_i \chi_{A_i} - \sum_{i=1}^n \alpha_i \chi_{h(A_i)} > 0.$$

(We first prove this theorem for integer coefficients, and then extend it to functions with rational and real coefficients.) Then $f - g > 0$ and $f - g \sim 0$. Since $f - g > 0$ and has positive integer values, we can define

$$\text{Gr}(f - g) = \{(x, r) \in G \times C_s : 0 < (f - g)(x) < r\} \subset G \times C_s.$$

Since $(f - g) \sim 0$ there exists a piecewise translation H in $G \times C_s$ such that $H(\text{Gr}(f - g)) = \text{Gr}(0) = \emptyset$. Then $\text{Gr}(f - g) = \emptyset$. This implies that $f - g = 0$.

If for some $x \in G$, $(f - g)(x) \neq 0$, then $(f - g)(x) > 0$ and $(x, (f - g)(x)) \in \text{Gr}(f - g)$. Thus $(f - g) = 0$, and since $f > 0$ and $-g > 0$, it follows that $f = 0 = g$ for all $x \in G$.

Now let us suppose that $f = \sum_{i=1}^n \alpha_i \chi_{A_i} > 0$ and $g = \sum_{i=1}^n \alpha_i \chi_{h(A_i)} < 0$, where the α_i 's are rational numbers. Let $D = \text{L.C.D.}\{\alpha_i\}$. Then Df and Dg satisfy the hypothesis of the case proven above and $Df = Dg = 0$. It follows that $f = g = 0$.

Next suppose that $f = \sum_{i=1}^n \alpha_i \chi_{A_i} > 0$ and $g = \sum_{i=1}^n \alpha_i \chi_{h(A_i)} < 0$, where $\alpha_i \in \mathbf{R}$.

Choose a Hamel basis C of \mathbf{R} over the rationals. Then each $\alpha_i = \sum_{c \in C} \alpha_i(c) \cdot c$, where $\alpha_i(c)$ is rational. Let $\{c_j : j = 1, \dots, m\}$ be the finite subset of C such that $\alpha_i(c_j) \neq 0$ for some $i = 1, 2, \dots, n$. Then $\alpha_i = \sum_{j=1}^m \alpha_i(c_j) \cdot c_j$, where $\alpha_i(c_j)$ is rational. Thus

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} = \sum_{i=1}^n \sum_{j=1}^m \alpha_i(c_j) \cdot c_j \chi_{A_i} > 0$$

and

$$g = \sum_{i=1}^n \alpha_i \chi_{h(A_i)} = \sum_{i=1}^n \sum_{j=1}^m \alpha_i(c_j) \cdot c_j \chi_{h(A_i)} < 0.$$

Let

$$f_j = \sum_{i=1}^n \alpha_i(c_j) \chi_{A_i} \quad \text{and} \quad g_j = \sum_{i=1}^n \alpha_i(c_j) \chi_{h(A_i)}.$$

Then

$$f = \sum_{j=1}^m c_j f_j \quad \text{and} \quad g = \sum_{j=1}^m c_j g_j.$$

Choose m monotonic decreasing sequences of rationals α_{jk} for $j = 1, 2, \dots, m$ such that $\lim_{k \rightarrow \infty} \alpha_{jk} = c_j$, for each j . Also choose m monotonic increasing sequences of rationals β_{jk} for $j = 1, 2, \dots, m$ such that $\lim_{k \rightarrow \infty} \beta_{jk} = c_j$.

Let

$$\bar{f}_k = \sum_{j=1}^m \alpha_{jk} f_j, \quad \bar{g}_k = \sum_{j=1}^m \alpha_{jk} g_j$$

and

$$\underline{f}_k = \sum_{j=1}^m \beta_{jk} f_j, \quad \underline{g}_k = \sum_{j=1}^m \beta_{jk} g_j.$$

Then $\bar{f}_k \geq f \geq \underline{f}_k$ and $\bar{g}_k \geq g \geq \underline{g}_k$ for every k .

Let ε_1 be the smallest nonzero value of f and let ε_2 be the absolute value of the largest nonzero value of g . Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, and choose k_0 such that $|\alpha_{kj} - c_j| < \varepsilon/mM$ and $|\beta_{kj} - c_j| < \varepsilon/mM$ for all $j = 1, 2, \dots, m$, for all $k > k_0$, and where $M = \max\{f_j, g_j : j = 1, \dots, m\}$. We know that $\bar{f}_k > f > 0$ and $\underline{g}_k < g < 0$ are true for all k .

For sufficiently large k we have $0 \leq f - \underline{f}_k < \varepsilon$ and $0 < \bar{g}_k - g \leq \varepsilon$.

If $f(x) > 0$, then $f(x) \geq \varepsilon$ and $\underline{f}_k(x) > 0$. If $f(x) = 0$, then since $f = \sum_{j=1}^m c_j f_j$, it is a consequence of the linear independence of the Hamel basis that $f_j(x) = 0$ for all $j = 1, 2, \dots, m$. But then

$$\underline{f}_k(x) = \sum_{j=1}^m \beta_{jk} f_j(x) = 0.$$

Thus $\underline{f}_k > 0$ for all $k > k_0$.

Similarly, if $g(x) < 0$, then $g(x) < -\varepsilon$ and $\bar{g}_k(x) < 0$. If $g(x) = 0$, then $\bar{g}_j(x) = 0$ for all j and

$$\bar{g}_k(x) = \sum_{j=1}^m \alpha_{jk} g_j(x) = 0.$$

Thus $\bar{g}_k(x) < 0$ for $k > k_0$.

It is easy to see from the definitions

$$f_j = \sum_{i=1}^n \alpha_i(c_j) \chi_{A_i} \quad \text{and} \quad g_j = \sum_{i=1}^n \alpha_i(c_j) \chi_{h(A_i)}$$

that $f_j \sim g_j$ for each $j = 1, 2, \dots, m$. It also follows from the definitions of \bar{f}_k , \bar{g}_k , \underline{f}_k , and \underline{g}_k as linear combinations of the f_j and g_j that $\bar{f}_k \sim \bar{g}_k$ and $\underline{f}_k \sim \underline{g}_k$. Since these functions are linear combinations with rational coefficients of

characteristic functions, the results of the preceding section of the proof apply and it follows that $\bar{f}_k = \bar{g}_k = 0$ and $\underline{f}_k = \underline{g}_k = 0$. Then $f = g = 0$.

This proves the first part of the theorem.

Now if

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} \geq 0 \quad \text{and} \quad g = \sum_{i=1}^n \alpha_i \chi_{h(A_i)} + \sum_{i=1}^m \beta_i \chi_{B_i}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ and the B_i are rational sets with $\mu(B_i) = 0$, we define f' and g' by $f'(x) = f(x)$ and $g'(x) = g(x)$ if $x \in G - \bigcup_{i=1}^m B_i$; and $f(x) = 0$ and $g(x) = 0$ if $x \in \bigcup_{i=1}^m B_i$. Then $f' \geq 0$ and $g' \leq 0$ and $f' \sim g'$. Thus $f' = g' = 0$, and $f = g = 0$ for all $x \notin \bigcup_{i=1}^m B_i$ and $\mu(\bigcup_{i=1}^m B_i) = 0$. \square

It follows immediately from Theorem 6.4 that if $f \geq 0$ and $g \leq 0$ and $f \approx g$, then $f \approx 0 \approx g$.

LEMMA 6.5. *The relation \geq is a partial ordering of V' .*

PROOF. (1) Reflexive. It is clear that $F \geq F$.

(2) Transitive. If $F_1 \geq F_2$ and $F_2 \geq F_3$, then $F_1 - F_2 \geq 0$ and $F_2 - F_3 \geq 0$. Then there exist $f \in F_1 - F_2$ and $g \in F_2 - F_3$ such that $f \geq 0$ and $g \geq 0$. Then $f + g \in (F_1 - F_2) + (F_2 - F_3) = F_1 - F_3$ and $f + g \geq 0$. Thus $F_1 \geq F_3$.

(3) Antisymmetric. If $F_1 \geq F_2$ and $F_2 \geq F_1$, then there exist $f \in F_1 - F_2$ and $g \in F_2 - F_1$ such that $f \geq 0$ and $g \geq 0$. Then $-g \in F_1 - F_2$ and $-g \leq 0$. Thus $f \approx (-g)$. Then it follows from Lemma 6.3 that $f \approx 0$, and thus $F_1 = F_2$.

DEFINITION 6.6. Let $M' = \{F \in V': F \cap M \neq \emptyset\}$. Then M' is a linear space. If $f_1 \in M \cap F_1$ and $f_2 \in M \cap F_2$, then $f_1 + f_2 \in M \cap (F_1 + F_2)$ and $af_1 \in M \cap aF_1$ for any constant a . Now we define a linear functional on M' by $L(F) = \lambda(f)$ for any $f \in F \cap M$. Then L is a nonnegative functional since λ is.

The next theorem, here for convenience, is adapted from Theorem 14 of Banach [1].

THEOREM 6.7. *If a nonnegative linear functional L_β is defined on the linear space $V_\beta \subset V'$ and $\varphi \in V' - V_\beta$ with $F_1 \geq \varphi \geq F_2$, F_1 and $F_2 \in V_\beta$, then L_β can be extended to a nonnegative linear functional $L_{\beta+1}$ defined on $V_{\beta+1} = (V_\beta, \varphi)$ such that $L_{\beta+1}(F) = L_\beta(F)$ if $F \in V_\beta$.*

PROOF. Let $\alpha = \inf\{L_\beta(F): F \in V_\beta \text{ and } F \geq \varphi\}$. Every coset in $V_{\beta+1}$ is of the form $F + c\varphi$. We can define $L_{\beta+1}(F + c\varphi) = L_\beta(F) + c\alpha$. It is clear that this is well-defined and linear. It remains to show that $L_{\beta+1}$ is nonnegative.

Suppose $F + c\varphi \geq 0$. If $c = 0$, then $L_{\beta+1}(F + 0 \cdot \varphi) = L_\beta(F) \geq 0$, since L_β is nonnegative.

If $c > 0$, then $\varphi \geq -F/c$ and for every $F_1 \geq 0$ we have $F_1 \geq -F/c$ or $F_1 + F/c \geq 0$. Thus $L_\beta(F_1 + F/c) \geq 0$ and $c\alpha + L_\beta(F) \geq 0$.

If $c < 0$, then $F/-c - \varphi \geq 0$. Thus $L_\beta(-F/c) \geq \alpha$ or $-L_\beta(F)/c \geq \alpha$, from which follows $L_\beta(F) + c\alpha \geq 0$, since $c < 0$.

Thus $L_{\beta+1}$ is a nonnegative linear functional on $V_{\beta+1}$. \square

THEOREM 6.8. *The nonnegative linear functional L defined on M' can be extended to a nonnegative linear functional L' defined on V' such that if $F \in M'$ then $L'(F) = L(F)$.*

PROOF. See Banach [1, Theorems 15 and 16].

Since every $f \in V$ is bounded and since all constant functions are in M , we know that for every $\varphi \in V'$ there are $F_1, F_2 \in M$ such that $F_1 \geq \varphi \geq F_2$. Thus we can apply the preceding theorem and transfinite induction to show that L can be extended to a nonnegative linear functional L' on V' . \square

It is now possible to define a nonnegative linear functional λ' on V such that if $f \in M$, then $\lambda'(f) = \lambda(f)$. We define $\lambda'(f) = L'(F)$, where $f \in F$.

Then λ' is a translation invariant nonnegative linear functional on V , i.e., λ' satisfies the following conditions:

- (1) $\lambda'(\chi_G) = 1$,
- (2) $\lambda'(af + bg) = a\lambda'(f) + b\lambda'(g)$,
- (3) $\lambda'(f) \geq 0$ if $f \geq 0$,
- (4) $\lambda'(\sum_{i=1}^n \alpha_i \chi_{h_i(A_i)}) = \lambda'(\sum_{i=1}^n \alpha_i \chi_{A_i})$, if the h_i are piecewise translations.

We can now easily define a Banach measure on G by defining $\mu(A) = \lambda'(\chi_A)$ for all $A \subset G$. Then it is obvious that μ satisfies the conditions:

- (1) $\mu(G) = 1$,
- (2) $\mu(A \cup B) = \mu(A) + \mu(B)$, if $A \cap B = \emptyset$,
- (3) $\mu(h(A)) = \mu(A)$ if h is a piecewise translation,
- (4) $\mu(A) \geq 0$ for all $A \subset G$.

Since the piecewise translation h in Condition 3 is a left piecewise translation, μ is a left translation invariant measure and G is left amenable. It is well known (see Day [5] and Greenleaf [6]) that if a group G is left amenable, then G is right amenable, and that if G is left and right amenable, then G is amenable. \square

This completes the proof of our main result.

THEOREM 6.9. *A group is amenable if and only if it does not contain a large paradoxical set.*

7. Comments and conjectures. The proof of Theorem 6.9 relies heavily on the use of $G \times C_n$ that is permitted by the definition of "large paradoxical set" as no large 2-into-1 in any $G \times C_n$. Clearly the theorem would be more satisfying if we could alter the definition of " G contains no large paradoxical

sets" to " G contains no large 2-into-1" and avoid mention of the direct product $G \times C_n$. This would also be a more natural generalization of Banach and Tarski's paradoxical sets. To do this and still have Theorem 6.9 be true it would be sufficient to prove the following conjecture:

CONJECTURE 7.1. Given a group G and a finite cyclic group C_n , G contains a large 2-into-1 if and only if $G \times C_n$ does.

To prove this, it suffices to show:

CONJECTURE 7.1'. If $G \times C_2$ contains a large 2-into-1, then so does G .

If it is true for C_2 then it is true for any C_n . The converse of Conjecture 7.1' is obvious.

The following conjectures are of considerable interest and closely related to this topic:

CONJECTURE 7.2. A group is amenable if and only if it does not contain a noncyclic free subgroup.

CONJECTURE 7.3. A group contains a large 2-into-1 if and only if it contains a noncyclic free subgroup.

In each of these conjectures one direction is easy. Thus we can rewrite them to show only the unknown direction:

CONJECTURE 7.2'. If a group is not amenable then it contains a noncyclic free subgroup.

CONJECTURE 7.3'. If a group contains a large 2-into-1 then it contains a noncyclic free subgroup.

Since the converse of Conjecture 7.3' is not trivial we include the proof.

THEOREM 7.4. *If a group contains a noncyclic free group, then it contains a large 2-into-1.*

PROOF. Let $F \doteq \langle a, b \rangle \subset G$. Then $G = \bigcup_{g \in E} Fg$. Elements of F can be written as words in a and b , $w = a^{n_1} b^{n_2} a^{n_3} \dots b^{n_k}$. Define piecewise translations ψ and φ as follows: if w begins in a negative power of a , then $\varphi w = a^{-1}w$; otherwise $\varphi w = aw$. If w begins in a negative power of b , then $\psi w = b^{-1}w$; otherwise $\psi w = bw$. Then $\varphi F \cap \psi F = \emptyset$, since every word of φF begins in a or a^{-1} and every word of ψF begins in b or b^{-1} . Then, since $G = \bigcup_{g \in E} Fg$, for each $x \in G$ we can write $x = wg$, where $w \in F$, and we can define φx and ψx for each $x \in G$ by $\varphi x = \varphi wg$ and $\psi x = \psi wg$. Then, since $\varphi F \subset F$ and $\psi F \subset F$ and $\varphi F \cap \psi F = \emptyset$, it is clear that $\varphi G \cap \psi G = \emptyset$. Thus since $\varphi G \cup \psi G \subset G$, G is a large 2-into-1. \square

The conjectures 7.2' and 7.3' are closely related.

THEOREM 7.5. *Conjecture 7.2' implies and is implied by Conjecture 7.3'.*

PROOF. Suppose G is not amenable. Then by Theorem 6.9, G contains a large paradoxical set. Thus $G \times C_n$ contains a large 2-into-1. Then according

to Conjecture 7.3', $G \times C_n$ contains a noncyclic free subgroup. But it is easy to see that if $G \times C_n$ contains a noncyclic free group then so does G . The other direction is obvious. \square

To show that there is in fact a distinction between groups which contain a paradoxical set and groups which contain a large paradoxical set, we present an example of a group which contains no *large* paradoxical set but does contain a 2-into-1. Let G be the group of isometries of the plane. Then G is amenable. The group K of translations of the plane contains the commutator subgroup of G . So G/K and K are abelian and therefore amenable. Thus by a theorem of von Neumann [3], G is amenable. It can be shown that G contains a free semigroup on two generators. We now quote Theorem 1.7 from Sherman [9]: the group G contains a 2-into-1 by translation if and only if G contains a free semigroup on two generators.

Thus there is a set $S \subset G$ such that $aS \cup bS \subset S$ and $aS \cap bS = \emptyset$ for some $a, b \in G$. Since the translations a, b are also piecewise translations, S is a 2-into-1.

Hence G is an amenable group which contains a 2-into-1. But from Theorem 6.9 we know it cannot contain a large 2-into-1. Therefore G contains a paradoxical set but no large paradoxical set.

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